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Discussione

On the Irreducibly Open Nature of Thought

In William Boos's *Metamathematics and the Philosophical Tradition*[*]

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A Description of the Book[1]

The title of the first chapter is "Introduction: Boundaries of Experience" (pp. 1-67), being a critical dialogue with the Aristotelian statement according to which all human beings aspire to knowledge. This reflection on the Aristotelian thesis leads to the development of a theory of knowledge whose main motivation consists in seeking to determine whether, by virtue of its very nature, knowledge can ever be complete. The important thing is to determine whether this process suggests the existence of one single truth or, on the contrary, its development is fundamentally divergent; this is where the author establishes some links with

[*] W. Boos, *Metamathematics and the Philosophical Tradition*, Berlin 2018.

[1] William Boos, the author of the book under review, was a philosopher and a mathematician whose main interest lay in the intersection of these two fields. Around the time of his premature death in 2014 the preparatory manuscript included twenty chapters and a possible appendix. The essays collected in this volume were selected by Florence Boos from this wealth of material. Fundamentally, the selection focused on the texts that dealt with classical, medieval and Enlightenment philosophy, thus making it possible to publish those articles which had remained unpublished. *Metamathematics and The Philosophical Tradition* is a collection of essays whose first eight chapters include an analysis of traditional philosophical problems from a *metamathematical* perspective. Chapter 9 focuses on the problematics of free will and determinism, while Chapter 10 deals with the time-evolution of theoretical physics. Formally, a great deal of care was visibly applied in the arrangement of the text so as to provide it with a neatly organised logical structure: the book is divided into chapters, with each chapter being divided into subchapters, and individual paragraphs themselves dovetailed with each other. The fact that the book is a collection of essays explains the occasional occurrence of repetitions or of some overlapping here and there, but this does not undermine the formal rigour of the book as a whole.

Kant's philosophy. Boos attributes an almost-theological dignity to this rational investigation or inquiry.

The above-mentioned opposition can also be described as a tension between a *hieratic* (tendentially dogmatic) and a *zetetic* (tendentially sceptical) conception of knowledge. The author sees the way how the two different views deal with the existence of logical paradoxes as an expression of such distinction: in the former case, paradoxes are but apparent and they can be eliminated through a correct interpretation of the problem. In contrast, for the author such paradoxes are a sign that the heuristic procedure can never be closed, that is, it can never have an end. As is clarified at a certain point in the book, the author believes that the hieratic conception is founded on a semantic monism according to which any theory devoid of a single interpretation is inconsistent. The author devotes more space to the discussion of certain ideas by Leibniz, Berkeley (whom he interprets as someone who harmonises a zetetic, non-sceptical conception with hieratic ideals) Hume (about whom he provides an original interpretation since he considers that this Scottish philosopher postulated a uniform, non-empirical absolute, which is *experience*), and Kant.

Boos considers the way in which acknowledging the distinction between hieratic and zetetic can influence not only epistemology but also ethics. From his zetetic standpoint, philosophy is understood as an activity that aims to study those problems whose solution has not been found by the science of a certain period. Therefore, the *horizon* of philosophy is, by nature, something in continuous movement as knowledge progresses and diversifies. Certain problems are initially regarded as strictly philosophical, since the way they can be solved is still unclear, and as scientific knowledge progresses, they become scientific problems.

The title of chapter 2 is ""Was Blind, But Now I See": Ramifications of Plato's Line" (pp. 68-104), being a reinterpretation of Plato's theory of forms according to the notion of *intended interpretation* or *model*. The author argues, as he generally does throughout the book, that a consequence of Gödel's incompleteness theorem is that every definition procedure is itself undefinable and always refers to a circumscribed context. This means that there may be a number of alternative definitions but each one of them is justifiable only within the metatheory that formulates it. However, such metatheory lacks the resources needed to prove the truth of that definition exclusively on its own basis. The Platonic forms would therefore be but provisional, idealised models capable of stimulating and enabling further inquiry. Each individual form would be the semantic referent of a syntactic entity representing it, always imperfectly, within a given theory.

The title of chapter 3 is "The Stoics, the Skeptics and Aporetic Autonomy: Is "What Is In Our Power" In Our Power?" (pp. 105-130). This chapter is fundamentally a presentation and an assessment of the debate between stoics and sceptics on different topics of philosophical interest, such as truth or, more precisely, an adequate criterion of truth. The author also identifies the distinction between *object-theory* and *metatheory* in some stoics, notably Cicero. According to Boos, there exists in Stoicism a tension between hieratic and zetetic elements. On the one hand, the stoics claim that that there is a unitary truth which encompasses the totality of temporal experience while, on the other, they cherish a notion of individual autonomy which is (in the author's opinion) challenged by the previous claim. The author closes this chapter by stating that what is truly *good*, in such a way as to be an end in itself, is the inquiring will itself in its search for knowledge.

The title of chapter 4 is "Anselm, *Fides Quaerens Interpretationem*, and *Grenzideen* as Generators of Metatheoretic Ascent" (pp. 131-159) and, as the title immediately suggests, it includes a critique of Anselm's argument. As claimed by the author, the argument fails in that it presupposes the consistency of a unique limit for conceptual inquiry. Therefore, the very attempt to define "*that than which nothing greater can be conceived*" (that is, the periphrastic name of God according to Anselm's conception) would generate paradoxes. Furthermore, for the author, in whatever theory in which the existence of such a singular limit was provable it would also be possible to prove the existence of any other desired result, which means that the metatheory itself would be inconsistent. In this chapter the author proposes to analyse Anselm's original argument and Gaunilo's criticism of it, examine the similarity in the formulation of the operative notion in Anselm and in Cantor's comparable notion, and reconsider Kant's critique of the argument as well as modern modal reconstructions of it.

One of the objections that the author directs at the argument in question is that it unjustifiably assumes that the limit to which the series converges is unique. This implies that the very notion of perfection invoked is ambiguous. Along the same line, the author argues that the argument presupposes that each series (since there is a multiplicity of series that tend towards a maximum) converges to the same maximal point which is equivalent to the maximal point of all other series, so that the maximum of a series is consistent only if the maxima of all the other series are equally consistent. For the author it is therefore impossible to identify the maximal point of the series since it is not even possible to clearly identify (define) each previous member of the series (for example, it is not possible to define exhaustively what each one of us is). The author keeps repeating this thesis: limited as we are, we are unable to clearly conceive of the maximal (and, therefore absolutely perfect) point of the series. The maximal point of the series would therefore be inexpressible and unintelligible "from beneath". (Thus, the thesis is the following: because the maximal point is understood as being the same for different series (as an aggregate – an expression that the author mentions at a certain point in this chapter – of everything thinkable), it is inconsistent.

The book under analysis is clearly an application of the consequences that can be extracted from Gödel's incompleteness theorems to some of the major dilemmas in the history of philosophy. Such thesis is clearly expressed on page 145. Boos's argumentation can be reconstructed as follows:

1. Every consistent and intelligible theory includes propositions that cannot be interpreted within that same theory.

2. Thus, no theory can provide a totalising interpretation of all its propositions

3. Therefore, there can be no theory (or metatheory) capable of providing a full explanation (and, consequently, offering itself as a fixed, stable point on which the intelligibility of all theories that it interprets is founded).

4. For this reason, there can be no ultimate, full realisation of that intelligibility.

The title of chapter 5 is "“Parfaits Miroirs de l'Univers”: A “Virtual” Interpretation of Leibnizian Metaphysics”" (pp. 160-185). It offers a metalogical reconstruction of certain aspects of Leibniz' metaphysics, such as identity, indiscernibility, the way how each being (monad) reflects the entire universe, metaphysical or absolute necessity, compossibility, the principle of sufficient reason, and perfection. The most curious aspect of this 5th chapter occurs precisely in the last pages, where the author notes that Leon Henkin discovered that a maximal consistent theory, U, sufficiently complete and perfect to decide upon the truth-value of all propositions in its language also secures the existence of a model (or interpretation) for itself. This result – which goes against the grain of the book's main thesis – is nonetheless, and according to the author, reconciled with it insofar as there is a plurality of interpretations, a number of which are indeed unintelligible.

The title of chapter 6 is “Berkeleyan Metalogical “Signs” and “Master Arguments”” (pp. 186-232). Following a recurrent pattern in the book, this chapter again includes an analysis and a metalogical reconstruction of a particular philosopher's thought – in this case, Berkeley, and especially his conception of general ideas, concept formation, his *master argument*, and his description of scientific inquiry. The opposition which, according to the author, exists between Berkeley's dialectical and semiotic inquiries and his ontological construction is of particular interest, especially since Berkeley claimed that the interpretation of any sign leads to the emergence of infinite metatheoretical hierarchies. As regards the book's main thesis, the key development in this chapter occurs towards the end, being exemplified through the following argument: all ultimate points of discernment require another, later moment that can apprehend or ratify them as such; therefore, it is not possible to guarantee any sense in which the existence of such points can be affirmed.

The title of chapter 7 is “The Second-order Idealism of David Hume” (pp. 233-305). In the very first pages of this chapter the author provides a surprising analysis according to which David Hume should not be considered a sceptic, but rather, according to his own expression, a second-order idealist, which would bring him closer to Berkeley. The common characteristic of first-order idealism and second-order idealism is the thesis according to which it is possible, in some circumstances, to deny that the existence of unperceived objects or indeterminate forces can be denied. This characteristic of Hume's thought stems from the thesis that conceivable objects are possible and may exist, while those that are unconceivable are impossible and cannot exist. This is a thesis which, according to the author, no sceptic can endorse. The most interesting aspect of Boos's thoughts on Hume emerges from the last sentence of the following text (I shall quote the full passage, although Boos does not):

> "Tho' the mind in its reasonings from causes or effects carries its view beyond those objects, which it sees or remembers, it must never lose sight of them entirely, nor reason merely upon its own ideas, without some mixture of impressions, or at least of ideas of the memory, which are equivalent to impressions. When we infer effects from causes, we must establish the existence of these causes; which we have only two ways of doing, either by an immediate perception of our memory or senses, or by an inference from other causes; which causes again we must ascertain in the same manner, either by a present impression, or by an inference from their causes, and so on, till we arrive at some object, which we see or remember. 'Tis impossible for us to carry on our inferences in infinitum; and the only thing, that can stop them, is an impression of the memory or senses, beyond which there is no room for doubt or enquiry"[2].

What Hume is saying is that all reasoning results from experience, whether by immediate action upon the senses or by the mediation of memory. It should be noted that Hume expressly states that there is an end to the whole process of doubt or inquiry, whose infinite extensibility he explicitly denies, namely, experience itself, which openly contradicts Boos's main thesis that, let us remember, such process is inextricably open.

The title of chapter 8 is "Kantian Ethics and "the Fate of Reason"" (pp. 306-381). To a certain extent, this chapter is a continuation of the previous chapter's discussion in that it clarifies the way how the author characterises what he means by *experience*. He begins by describing the main characteristics of experience: (i) the notion of experience is a rational, though problematic regulative ideal; (ii) the only unity to which experience can aspire is that of an intelligible process of inquiry, which is nonetheless essentially incomplete, and (iii) almost all maximal limits towards which this process of inquiry converges are intrinsically unintelligible. As for the key argument of the book, the main development of this 8th chapter corresponds to the thesis that the existence of a unique point of discernment is even more unsustainable within an ethical or practical dimension than within a scientific dimension. The reason is that the former is even more complex than the latter.

Chapter 9 is titled "Metamathematical Interpretations of Free Will and Determinism" (pp. 382-405), and the main thesis here is that the physical evolution of the universe is not governed by continuous, deterministic principles, since it is ramified, incomplete, and indeterminate. The author starts by quoting three different authorities from the past, Leibniz, Boscovich, and Laplace, who

[2] T 1.3.4.1, SBN 82-3.

voiced the exact opposite thesis. For Boos, on the contrary, physical-mathematical theories should be considered intelligible, though incomplete, and therefore, the time-evolution of the universe which they describe always occurs in a ramified, discontinuous way.

The title of chapter 10, the last chapter in the book, is "Time-Evolution in Random "Universes"" (pp. 406-441). The essay is in some way unique when compared to the others insofar as it departs the furthest from strictly philosophical considerations to foray more distinctly into a different area, that of Physics. It is also the most demanding chapter as concerns the readers' previous knowledge of both Mathematics and Physics. The core problem discussed in this chapter is the time-evolution of quantic systems and the thesis is sustained that the time-evolution of the universe is a stochastic or a random process.

On the Consequences of Gödel's Theorems for Natural Theology

The aim of this section is to introduce a brief discussion concerning two different moments in Boos's argument. The first is his critique of the ontological argument and the second is the possibility of extracting the conclusion that the principle of sufficient reason (PSR) is false, based on Gödel's incompleteness theorems. Naturally, this discussion will necessarily be carried out from a specific standpoint which, using the book's own language, Boos would perhaps consider *hieratic*. I nevertheless believe that, as is widely recognised, the best way to honour a thinker is to approach his/her ideas critically, which is particularly true when his/her thought differs from our own, this difference being exactly what encourages one to (re)think in an increasingly demanding and rigorous manner. "*Amicus Plato, sed magis amica veritas*", as the Romans used to say.

Anselm's Argument

The fact that Gödel himself advanced a version of the ontological argument is extremely intriguing (and it certainly must unsettle us). The episodes surrounding this argument are indeed delicious and they could certainly belong in a novel. Gödel worked on his argument for years and there is a partial version of it in a text that dates back to 1941[3]. In 1970, convinced that he was about to die, Gödel revealed his demonstration to his student Dana Scott and, although he died in 1978, the truth is that he never came to publish his article. The draft of his demonstration consists in a couple of pages with notes which were kept by Dana Scott.[4] The argument was revised and corrected by C. Anthony Anderson after Gödel's original version had been critiqued by J. Howard Sobel,[5] and it

[3] Cf. M. Fitting, *Types, Tableaus, and Gödel's God*, Dordrecht 2002, p. 138.
[4] Cf. Robert C. Koons, *Sobel on Gödel's Ontological Proof*, «Philosophia Christi», 8, 2006, p. 1.
[5] C. A. Anderson, *Some Emendations Of Gödel's Ontological Proof*, «Faith and Philosophy: Journal of the Society of Christian Philosophers». 7, 3, art. 3, 1990, pp. 291-303.

was later revised.[6] More recently, authors like Alexander Pruss,[7] Petr Hájek,[8] and A. P. Hazen[9] have dedicated some attention to it and the product of different reflections on Gödel's argument has been published in a number of books.[10]

The aspect highlighted in the above paragraph will not be analysed here, that is, I will not be introducing or discussing Gödel's specific version of the argument. I thought that it should mention it just to note that the very author of the formal result invoked by Boos to de-articulate the argument endeavoured to produce a valid version of the same family of arguments. What I meant to do was simply to raise the suspicion that the generic form used in any version of the ontological argument can resist Gödel's incompleteness theorems.

More substantively, my aim is to draft a brief response to a criticism of Gödel's argument made by Boos. He argues that the maximal limit for which the series considered in the argument converges is inconsistent. In order to address this criticism two movements are needed: the first explaining the way how the conception of such limit is reached, and the second concerning the very formulation of this limit and the elucidation of the question of whether or not it is consistent.

As regards the first question, and following Maria Leonor L. O. Xavier,[11] I will argue that the Anselmian notion of God is an *a posteriori* conception, so much so that it is constructed by using all the resources available to discursive reason. Notwithstanding the fact that the Anselmian periphrasis is a negative name of God[12] and God is therefore presented as an absolute, all the order relations used to obtain the notion in question are precisely notions of the order of the thinkable. Consequently, God is thinkable because he possible, i.e., the notion of God does not involve a contradiction. It is in his answer to Gaunilo that Anselm most clearly expounds the way how his understanding of God is obtained, namely, through an operation of thought by denying the limitations inherent in the beings that are present in our experience of the world. It should

[6] C. A. Anderson and M. Gettings, *Gödel's Ontological Proof Revisited*, in «Lecture Notes in Logic 6», ed. by Petr Hájek, Berlin 1996, pp. 167-172.

[7] A. R. Pruss, *A Gödelian ontological argument improved*, in «Religious Studies», 45, 2009, pp. 347-353.

[8] P. Hájek, *Magari and others on Gödel's ontological proof*, in A. Ursini and P. Aglianò (eds.), *Logic and Algebra*, New York 1996, p. 125.

[9] A. P. Hazen, *On Gödel's ontological proof*, «Australasian Journal of Philosophy», 76, no. 3, 1998, pp. 361-377.

[10] For example, *Ontological Proofs Today*, edited by Mirosław Szatkowski, which includes a number of articles that discuss Gödel's argument, and Pruss, who was mentioned above and who, in collaboration with Joshua Rasmussen, wrote the book *Necessary Existence* in which the same argument is, again, analysed.

[11] Cf. M. L. L. O. Xavier, *Do Pensável e do Impensável na filosofia do Argumento Anselmiano*, «Revista Portuguesa de Filosofia» 64, 1, 2008, p. 279.

[12] It is negative because it implies a double negation, i.e., an explicit negation and an implicit negation. The explicit negation is the one that refers to all higher terms: the periphrasis *that than which nothing greater can be thought* explicitly negates the relation with each and every higher term. The implicit negation is the relation with all lower terms of the series: God is not simply conceived of as being the mere maximal term of the series.

be noted that these beings which belong to our worldly experience not only start to exist and cease to exist, but also that they exist unevenly, to the rhythm of each instant that comes and goes. To think the existence of he *than which nothing greater can be thought* is, therefore, *to think* his existence as overcoming all time and space limits, but it is nevertheless *to think it.* If it is thought, it can therefore be conceived of and, this being the case, then it is not contradictory. Resorting to the epistemological principle according to which imagination is a guide for possibility (and, if this is true, that if a scenario can be imagined in considerable detail with no evident level of absurdity, then we have good reasons to argue that it represents a metaphysical possibility),[13] we can say that it is really possible.

As for the second aspect, it should be noted that not only does the notion *he than which nothing greater can be thought* not appear to be contradictory but it is also formalisable.[14] Not only does this possibility emerge as yet another confirmation that such limit is indeed possible but its formalisation does not appear to present any formal problems (i.e., the formula is neither malformed nor contradictory). Consequently, it is a possible, that is, a consistent, notion.

The Principle of Sufficient Reason

It thus seems that Anselm's argument has the ability to resist the challenges posed by Gödel's incompleteness theorems. Indeed, if we consider William Boos's argument in page 145, the argument that seems to be the most vulnerable to such challenges is, using Edward Feser's terminology,[15] the *rationalist proof*, which is, fundamentally, the Leibnizian version of the cosmological argument, because it explicitly uses the principle of sufficient reason. The different Leibnizian arguments are generically formulated as follows:

1. Every contingent fact has an explanation.
2. There is a contingent fact that includes all other contingent facts.
3. Therefore, there is an explanation of this fact.
4. This explanation must involve a necessary being.

[13] Cf. R. C. Koons and T. Pickavance, *The Atlas of Reality – A Comprehensive Guide on Metaphysics*, West Sussex 2017, p. 102.

[14] There are numerous formalisations of the argument, although naturally, some are easier to understand than others. As for the periphrasis *he than which nothing greater can be thought*, I must point out that the simplest formulation that I know of is the one by Paul Oppenheimer and Edward Zalta in the article "On the Logic of the Ontological Argument". This formalisation is the following: $*x*(*Cx* & Ø$*y(Gyx* & *Cy*)). That is, there is a conceivable object which is such that nothing greater can be conceived. The author of this review would venture to further refine the previous formulation by emphasising the notion that an object different from the first that is greater than it cannot be conceived. It thus follows that: $*x*(*Cx* & Ø$*y*((*Gyx* & *y*¹*x*) & *Cy*)).

[15] Cf. E. Feser, *Five Proofs of the Existence of God*, San Francisco 2017, p. 147.

5. This necessary being is God.[16]

Although the PSR may assume different, stronger or weaker, forms a generic formulation of this principle can be: *everything that is the case must have a reason why it is the case*. Thus, every true proposition or, at least, every contingent true proposition has an explanation.[17] The problem with Gödel's incompleteness theorems for the PSR is the fact that it raises a suspicion that the PSR is false when applied to mathematical propositions that are true but whose truth is not provable. This being so, proposition number (1) of the above argument is dismissed, and therefore the argument cannot continue.

A possible argument would be that the PSR is legitimately used when applied to metaphysical questions and even if it be demonstrated that the PSR is not a necessary principle logically, this does not invalidate the previous conclusion. Gödel's incompleteness theorems indicate that a formal proof of the truth of a proposition, such that it would follow from the axioms of the system through the application of the rules of transformation, is not the only reason why this proposition is true.

Conclusion

As becomes clear from this reading of Boos's book, we are in the presence of an extremely rich though also an extremely demanding text. Readers should ideally be somewhat conversant with metalogic to be able to venture fruitfully into its pages; I would also add that for someone who, either for professional reasons or merely out of curiosity, may need to investigate into the impact of Gödel's theorem on epistemology, this book is mandatory reading.

Appendix

Gödel's Theorem

I would now like to elucidate the nature of Boos's book while specifying what is meant by *metamathematics* or *metalogic*. In order to do that, it may be useful to start by clarifying what the book *is not*: it is not a reconstruction of different arguments in the history of philosophy using the tools of formal logic, be it propositional calculation, the calculation of predicates, or modal logic. As the tittle immediately makes clear, Boos's book is about *metamathematics*. *Metamathematics* can be defined as the branch of mathematics that provides

[16] Cf. A. R. Pruss, *The Leibnizian Cosmological Argument*, in W. L.Craig and J. P. Moreland (eds.), *The Blackwell Companion to Natural Theology*, West Sussex, 2009, pp. 25-26.
[17] Cf. A. R. Pruss, *The Principle of Sufficient Reason – A Reassessment*, New York 2006, p. 3.

proofs of what can be demonstrated in the discipline[18] and, as such, seeks to establish the consistency of mathematics itself.[19]

It may also relevant to mention some key notions pertaining to the study of formal languages and their interpretation. A formal language is a set of symbols (which includes relation, function, and constant symbols).[20] A model for a given language is a pair (A,I), where A corresponds to a collection of symbols and I corresponds to the interpretive function, linking symbols to language relations, functions or constants. A formula where no free variable occurs is a proposition. A set of propositions is consistent when there is no proof in which it is possible to derive a contradiction from that set. A formal system is complete when the set of theorems of this formal system coincides with the propositions that are true in this same system. If a proposition is true, then it is provable, and if it is provable, then it is true. A set is decidable when there is a procedure (that is, an algorithm) that makes it possible to determine whether a given proposition is a theorem (that is, a formula that can be derived from the axioms) of the system.

What the different articles included in the book do is develop the consequences of a given metamathematical result and apply it to the interpretation of some key issues in the history of philosophy, the result being Gödel's incompleteness theorems. The author evades the discussion of what Gödel's incompleteness theorems ultimately are and he seems to rely on at least some previous understanding of this result on the part of his readers. Therefore, I shall now attempt a (largely informal) description of Gödel's incompleteness theorems.

The main result of Gödel's incompleteness theorems is: arithmetic cannot be reduced to any axiomatic system. An axiomatic system is characterized as follows: in an axiomatic system there must exist a finite procedure by means of which it can be checked whether a sequence of symbols is an axiom, an inference rule, or a proof. [21] An axiom is a proposition of a formal system which is not derivable within that system from any other proposition. A theorem is a proposition resulting from axioms through the application of inference rules. In an axiomatic system all propositions which are taken as axioms are true and all propositions which are derivable from those axioms are also true.

What Gödel's incompleteness theorem demonstrates is that it is impossible to reduce arithmetic to an axiomatic system. The reason is that one of the two following results will occur: either (i) some arithmetic truths are unprovable, or (ii) some arithmetic falsities are provable. That is, there are either truths that are not provable or falsities that are provable.

[18] Cf. G. S. Boolos, J. P. Burgess and R. C. Jeffrey, *Computability and Logic, Fifth Edition*, New York 2007, p. 215.
[19] Cf. S. C. Kleene, *Introduction to Metamathematics*, New York 1971, p. 59.
[20] Cf. T. Jech, *Set Theory – The Third Millennium Edition, revised and expanded*, New York 2002, p. 155.
[21] Cf. H. J. Gensler, *Godel's Theorem Simplified*, New York 1984, p. 2.

It is nonetheless true that some parts of arithmetic are reducible to an axiomatic system, as is the case with the part that exclusively includes the operation of addition. The arithmetic of addition is an axiomatic system. This is demonstrable insofar as it can be reduced to a system which includes an axiom and two rules of inference. Any true well-formed formula in this system can be derived by using this axiom and these two rules of inference, but no false formula in this system is derivable. All particular cases of axioms have the form "$x = x$" and the two rules of inference may be used to simplify any term to its equivalent numeral. It should be noted that in this system we do not even need to know the meaning of the symbols in order to be able to apply the rules of transformation and derive the results.

The important thing is that in the arithmetic of addition all true well-formed formulas can be simplified to an equivalence between simple identities. This implies that all true well-formed formulas can be proven to be true. In other words, if a formula is true, then it can be proven. The reverse also applies: if a formula can be proven, then it is true. Thus, there is an equivalence between a formula being true and a formula being provable: every formula is true if, and only if, it is provable. Consequently, the arithmetic of addition can be reduced to an axiomatic system.[22]

I shall now consider another system, which contains multiplication and exponentiation. This system can be reduced to an axiomatic system too. This implies that the proofs of well-formed formulas can be mechanically created from simple identities and therefore, all true well-formed formulas are provable in the formal system under consideration. Again, and similarly, results that are provable are also true. In this arithmetic system, therefore, there is a certain number of propositions which are assumed as being the system's axioms and they are all true. Likewise, the transformation rules used are valid since necessarily true conclusions are drawn from them when applied to true premises.

The next system to be considered results from the combination of the two previous ones, comprising not only the operations of addition, multiplication and exponentiation but also symbols that express logical variables and constants. Among the latter we can find connectors such as negation ("it is not the case that …"), conjunction ("… and _ _ _"), a disjunction ("… or _ _ _") and implication ("if …, then _ _ _"), and quantifiers: the universal quantifier (" for every …") and the existential quantifier ("for some …"). It is with regard to this system that Gödel's incompleteness theorem establishes the result that it cannot be reduced to an axiomatic system. The reason for this impossibility resides in the fact that any attempt to axiomatise this system leads to the result that either some arithmetic truths are unprovable or that some arithmetic falsities are provable.

We should note that first-order logic, that is, the logic whose vocabulary includes propositional letters, connectives, individual variables, individual

[22] Cf. H. J. Gensler, *Godel's Theorem Simplified*, New York 1984, p. 15.

constants, predicates, relations, and quantifiers (the sphere of quantifiers including only individual variables, not properties) is an axiomatic system, a result which is proven by Gödel himself.[23] However, when first-order logic and the arithmetic system described above are jointly considered, then the resulting system is not reducible to an axiomatic system. This means that in this joint system there would be arithmetical truths which would be provable only if extra axioms were added to the system. However, if this were done other true propositions would emerge, although their truth would be provable only if even more axioms were added, and so on and so forth. According to Gödel's theorem there are always arithmetical truths that can only be proven if further axioms are added to our system. This picture becomes even more worrying when we realise that in the resulting system there are indeed truths that are unprovable or, alternatively, that there are falsities which are provable.

The problem becomes apparent when self-referential propositions are identified within the system, stating something like '*this proposition is not provable*'.[24] This result stems from the self-referentiality of certain arithmetic systems. Consequently, what is true is not the same as what is provable. In the case of arithmetic what happens is that every arithmetic system can be part of a stronger system that contains it in such a way that every arithmetical truth that was provable in the former system is provable in the latter, besides also other truths which the original system was unable to prove. However, there is no final, complete system, only a succession of progressively stronger systems.[25]

These results are summarised in Gödel's incompleteness theorems. The first theorem states that:

Every consistent axiomatisation of arithmetic contains an undecidable formula

In order to understand what this means we must remember that a system is '*consistent*' when it does not include a formula and its negation as theorems. A formula is '*undecidable*' in a system when neither this formula nor its negation is a *theorem* of the system (that is, they are provable from the system's axioms). Therefore, an alternative formulation of the first incompleteness theorem is:

Every consistent formalisation of arithmetic is incomplete

Since, given the principle of bivalence, either the undecidable formula or its negation must be true (although neither of the two is a theorem) we may conclude that one of the corollaries of Gödel's theorem is that:

Every consistent axiomatisation of arithmetic contains truths which are unprovable within the system in question

[23] Cf. H. J. Gensler, *Godel's Theorem Simplified*, New York 1984, pp. 37-38.
[24] *Ibidem*, p. 41.
[25] *Ibidem*, p. 60.

Gödel's second incompleteness theorem can be expressed as follows:

If an axiomatisation of arithmetic is consistent, then the proposition that formalises the information that 'This system is consistent' is not itself provable within the system.